Course Modules and Study Units

Module 1

The
Cartesian
Plane
and
Functions

UNIT 1

Preparation

Topics

Unit 1 covers the following topics:

- graphing polynomial and rational equations on a coordinate system using algebra
- finding equations of lines in point-slope form, slope-intercept form, and standard form
- function terminology: *domain*, *range*, *extrema*, *intercepts*, *asymptotes*, and *inverse*
- working with transcendental functions: trigonometric functions, inverse trigonometric functions, logarithmic and exponential functions

Textbook Assignment

Study the Preliminaries chapter (chapter 0), sections 0.1 (pp. 6–16), 0.2, 0.3, 0.4, and 0.5 (through example 5.11) in the course textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

The Meanings of *f*(*x*)

Make sure that you distinguish between the meaning of $f(x)$ in algebra and the meaning of $f(x)$ in functional notation. In the former, $f(x)$ means f times x; in the latter, $f(x)$ expresses a rule that associates each number in the domain with a corresponding number in the range. For example, $f(x) = x^2$ means that with a number, say 3, we associate the square of that number, 9, and collect the two numbers as an ordered pair (3,9).

Domain

If the domain is not given to you, then you are to assume it is the largest set of real numbers that makes sense. Since we are dealing only with real-valued functions in this text, there are only two places where you cannot use all real numbers for the most part as the domain:

1. If there is a denominator, we must not include any real number that makes the denominator equal to zero. For example, if for

$$f(x) = \frac{2x+1}{x-3}$$

   we set $x–3 = 0$ and solve for x, then $x = 3$ makes the denominator zero, and the domain of f is the set of all real numbers except 3. We can conveniently write this as $x \neq 3$.

2. You cannot have a negative number under a square root or, in general, under any even root. Thus, if

$$f(x) = \sqrt{2x-5}$$

   or

$$f(x) = \sqrt[4]{2x-5}$$

   for example, then we set $2x - 5 \geq 0$, and, solving for x, we have as the domain all $x \geq 5/2$.

The Expression $\log_a x$

Read the expression $\log_a x$ as "the power that you have to raise a in order to get x." Likewise, read $\sin^{-1} x$ as "the angle whose sine is x."

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 0.1

Do exercises 19, 29, 43, 49, 63, and 65 on pages 16–18 of the textbook.

Section 0.2

Do exercises 1, 3, and 17 on pages 24–25 of the textbook.

Section 0.3

Do exercise 11 on page 29 of the textbook.

Section 0.4

Do exercises 1, 3, 37, 43, and 69 on pages 40–41 of the textbook.

Section 0.5

Do exercises 1, 5, 7, 41, and 45 on pages 51–52 of the textbook.

MODULE 1 EXERCISES

Written Assignment 1

Complete the following exercises based on assigned sections you learned in this module, and submit them to your mentor for correction and grading. Show all calculations.

Note: *To facilitate assignment preparation and save you time typing, we have attached an assignment sheet, with all questions typed out for you in advance, to each assignment link in the Submit Assignments area of the course Web site. You can download these sheets to your computer and use them to insert your answers and submit them to your mentor. The assignment sheets are in rich text format and require MathType.*

1. **Find a second point on the line with slope *m* and point *P*, graph the line, and find an equation of the line.**

$$m = -\frac{1}{4}, \quad P = (-2,1)$$

2. **Find an equation of a line through the given point and *(a)* parallel to and *(b)* perpendicular to the given line.**

$y = 3(x-2)+1$ at (0, 3)

3. **Find the domain of the function.**

$$f(x) = \sqrt[3]{x-1}$$

4. **Find the indicated function values.**

$$f(x) = \frac{3}{x}; \quad f(1),\ f(10),\ f(100),\ f(1/3)$$

5. **Find all intercepts of the given graph.**

$$y = x^2 + 4x + 4$$

6. **Factor and/or use the quadratic formula to find all zeros of the given function.**

$$f(x) = x^2 + x - 12$$

7. Sketch a graph of the function showing all extrema, intercepts, and asymptotes.

(a) $f(x) = 3 - x^2$

(b) $f(x) = -x^2 + 20x - 11$

8. Sketch a graph of the function showing all extrema, intercepts, and asymptotes.

(a) $f(x) = 10 - x^3$

(b) $f(x) = -x^3 + 30x - 1$

9. Find all vertical asymptotes.

$$f(x) = \frac{x+4}{x^2 - 9}$$

10. Determine whether the function has an inverse (is one-to-one). If so, find the inverse and graph both the function and its inverse.

$$f(x) = \sqrt{x^2 + 1}$$

11. Convert the given radians measure to degrees.

(a) $\frac{3\pi}{5}$

(b) $\frac{\pi}{7}$

(c) 2

(d) 3

12. Convert the given degrees measure to radians.

(a) 40°

(b) 80°

(c) 450°

(d) 390°

13. Evaluate the inverse function by sketching a unit circle, locating the correct angle, and evaluating the ordered pair on the circle.

$\tan^{-1} 0$

14. Evaluate the inverse function by sketching a unit circle, locating the correct angle, and evaluating the ordered pair on the circle.

$\csc^{-1} 2$

15. A person who is 6 feet tall stands 4 feet from the base of a light pole and casts a 2-foot-long shadow. How tall is the light pole?

16. Convert the exponential expression into fractional or root form.

4^{-2}

17. Convert the exponential expression into fractional or root form.

$6^{2/5}$

18. Convert into exponential form.

$\frac{4}{x^2}$

19. Rewrite the expression as a single logarithm.

$2\ln 4 - \ln 3$

20. Rewrite the expression as a single logarithm.

$\ln 9 - 2\ln 3$

Module 2

Limits and Continuity

UNIT 2

Limits

Topics

Unit 2 covers the following topics:

- obtaining and using an intuitive definition of the concept of a limit
- applying the epsilon delta definition of a limit to simple functions
- using the basic limit rules to analytically compute limits of more complicated functions
- using factoring, rationalizing, and other algebraic techniques to compute analytically limits of various types of functions
- identifying the limit of a function at various points by using its graph
- applying key limit theorems to compute a limit of a function
- using a calculator for determining limits when all else fails

Textbook Assignment

Study sections 1.2 and 1.3 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Limits

The easiest way to view a limit is first to consider an infinite sequence of numbers, for example,

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \ldots \frac{1}{n}, \ldots$$

and to ask, If we keep going further and further in the sequence, do these numbers get closer and closer to a particular number without necessarily ever getting there, or having gotten there, never depart from that number? If so, then that number is called the limit of the sequence.

In the example given above, the numbers get closer to 0. We then say,

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

Likewise,

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \frac{n}{(n+1)} \ldots$$

has a general term

$$a_n = \frac{n}{(n+1)}$$

and gets closer and closer to 1. Thus,

$$\lim_{n \to \infty} \frac{n}{(n+1)} = 1.$$

Notice that even though the numbers never get to where they are going, we are able to determine what that number is. We have the ability to know what the ideal situation is and whether we are getting closer and closer to it even though we know we can never reach it. Possessing this ability makes the whole subject of calculus not only possible but practical.

Sequences like 1,2,3,4,5... do not tend to a number but rather get larger and larger without bounds. Even though technically there is no limit, we say

$$\lim_{n\to\infty} n = \infty.$$

This gives us more information than simply saying the limit does not exist. For a sequence like $1,-1,1,-1,1,-1,\ldots(-1)^{n+1}\ldots,$ the limit does not exist. It does not tend to ∞ but rather bounces back and forth between 1 and –1 and approaches no single number.

Sequences with limits do not have to tend to the limits in any orderly fashion. Thus,

$$1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4}\ldots$$

still approaches 0 as $n \to \infty$.

Furthermore, no one cares about the first trillion terms of a sequence. Only the infinite part determines if a limit exists. Thus,

$$2,119,-1062,58,\frac{2}{3},6,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots$$

still tends to 0. The sequence 1,1,1,1,1,… tends to 1 as a limit. Actually, it never gets off 1, but due to the formal definition of limit, we regard 1 as the limit of this sequence.

Limits of Functions

In our text we consider limits of functions. For example, consider

$$\lim_{x\to 3}(2x+1).$$

Do we interpret this to mean that as x gets closer and closer to 3 without ever getting there, $2x+1$ gets closer and closer to a particular number?

The answer is yes. As $x \to 3$, $(2x + 1) \to 7$. In fact, if you replace x in the function by where it is going and you get an honest to goodness number, that number will be your limit. But keep in mind that you have just committed an illegal operation, since x was never supposed to reach 3. Thus,

$$\lim_{x\to 5}(x^2 - 4) = 21$$

and it would seem that taking limits of functions is trivial.

Consider, however,

$$\lim_{x\to 3}\frac{(x^2-x-6)}{(x-3)}.$$

If we replace x by 3 in the expression, we get 0/0. This is not a number. It is called an indeterminate, and it means you have no idea what the limit is. In fact, you can conclude nothing about the limit from an indeterminate. When you get 0/0 or ∞/∞, it means that you have work to do to find out what the limit is. In this case, what you do is factor the numerator, that is,

$$\lim_{x\to 3}\frac{(x^2-x-6)}{(x-3)}=\lim_{x\to 3}\frac{(x-3)(x+2)}{(x-3)}=\lim_{x\to 3}(x+2)=5.$$

So the subject is not trivial and requires a variety of methods to go from an indeterminate to a form where the limit is obvious. As another example, consider

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}.$$

Once again replacing x by 1 gives us 0/0. In this case we rationalize the numerator and write

$$\frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\frac{x-1}{(x-1)(\sqrt{x}+1)}=\frac{1}{\sqrt{x}+1}\to\frac{1}{2}.$$

This is allowable, since all we have done is to multiply the function by 1, which does not change the function. In fact, the only two things that do not change an expression are multiplying by 1 or adding 0.

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 1.2

Do exercises 5 and 7 on page 76 of the textbook.

Section 1.3

Do exercises 5, 9, 13, 21, 27, 33, and 37 on page 85 of the textbook.

UNIT 3

Continuity, One-Sided Limits, and Infinite Limits

Topics

Unit 3 covers the following topics:

- using one-sided limits when necessary to determine whether a limit exists
- applying the definition of continuity to determine whether a function is continuous at a particular point and, if discontinuous, to identify the type of discontinuity
- using the properties of a continuous function in conjunction with the Intermediate Value Theorem to help find the roots (zeros) of a polynomial function
- using infinite limit to establish vertical asymptotes of rational and transcendental functions

Textbook Assignment

Study sections 1.4, 1.5, and 1.6 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Continuity

The definition of a continuous function at a, value $x = a$, requires three conditions:

a. $f(a)$ exists.

b. $\lim_{x \to a} f(x)$ exists.

c. $\lim_{x \to a} f(x) = f(a)$, which means that the function is defined at a and is equal to the limit as x approaches a.

If one of the conditions is violated, then the function is discontinuous at a. We can classify the type of discontinuity we are dealing with based on which condition is not satisfied. But first it is useful to know that if $f(x)$ is a polynomial, then it is continuous everywhere.

Removable Discontinuities If condition *(b)* is satisfied, then if $f(x)$ is not continuous at $x = a$, it is a **removable discontinuity,** since if $f(a) \neq \lim_{x \to a} f(x)$ or $f(a)$ is not defined, then we can define or redefine it to be equal to $\lim_{x \to a} f(x)$ and in effect remove the discontinuity.

Jump Discontinuities If $\lim_{x \to a} f(x)$ does not exist, but

$$\lim_{x \to a^-} f(x) \text{ and } \lim_{x \to a^+} f(x)$$

both exist (i.e., the left- and right-hand limits exist but have different values), then we have a **jump discontinuity.**

General Discontinuities If either the left- or right-hand limit does not exist (or is infinite), then we have a **general discontinuity.**

To illustrate these three types of discontinuities, consider the following examples. In each case we consider the continuity of the function at $x = 2$.

EXAMPLE 1

Let $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 5, & x = 2 \end{cases}$

a. $f(2) = 5$

b. $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \left(\dfrac{x^2 - 4}{x - 2} \right) = \lim\limits_{x \to 2} \dfrac{(x-2)(x+2)}{(x-2)} = 4$

c. $5 \neq 4,$ therefore the function is discontinuous at $x = 2.$

However, if we define $f(2) = 4$ instead of 5, then the function will be continuous. Therefore, $x = 2$ is a **removable discontinuity.**

EXAMPLE 2

Let $g(x) = \begin{cases} 2x + 1, & x \leq 2 \\ x^2 + 2 & x \geq 2 \end{cases}$

a. $g(2) = 2(2) + 1 = 5$

b. $\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^-} (2x + 1) = 5$

$\lim\limits_{x \to 2^+} g(x) = \lim\limits_{x \to 2^+} (x^2 + 2) = 6$

c. Since $5 \neq 6,$ no limit exists, and we have a **jump discontinuity.**

EXAMPLE 3

Let $h(x) = \dfrac{1}{(x-2)}$

a. $h(2) = \dfrac{1}{(2-2)} = \dfrac{1}{0}$, which is undefined.

b. $\lim_{x \to 2^-} \dfrac{1}{(x-2)} = -\infty$

Therefore, the function is discontinuous at $x = 2$ with a **general discontinuity.**

Infinite Limits

In handling infinite limits, consider the following:

$$\lim_{x \to 0} \frac{1}{x}.$$

If we replace *x* by 0, we get 1/0. This is not an indeterminate. Rather, if the denominator of a fraction gets very close to 0 and the numerator stays fixed, the fraction numerically becomes larger and larger. Thus,

$$\left|\frac{1}{0}\right| = \infty.$$

However, we do not know whether it tends to ∞ or $-\infty$ or neither. For this we need one-sided limits.

$$\lim_{x \to 0^-} \left(\frac{1}{x}\right) = -\infty, \text{ since } x < 0$$

and

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right) = +\infty, \text{ since } x > 0$$

Since the left- and right-hand limits tend toward different infinities, there is no limit.

However,

$$\lim_{x\to 0^-}\left(\frac{1}{x^2}\right)=\infty$$

and

$$\lim_{x\to 0^+}\left(\frac{1}{x^2}\right)=\infty$$

since $x^2>0$ for positive or negative *x*. Thus, we can say that

$$\lim_{x\to 0}\left(\frac{1}{x^2}\right)=\infty.$$

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 1.4

Do exercises 3, 7, 11, 21, and 37 on pages 94–95 of the textbook.

Section 1.5

Do exercises 7, 9, 15, 29, and 67 on pages 103–105 of the textbook.

Section 1.6

Do exercise 3 on page 116 of the textbook.

MODULE 2 EXERCISES

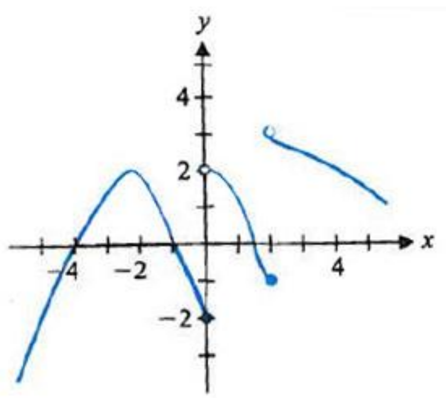
Written Assignment 2

Complete the following exercises based on assigned sections you learned in this module, and submit them to your mentor for correction and grading. Show all calculations.

1. Use numerical and graphical evidence to conjecture values for the following limit. If possible, use factoring to verify your conjecture.

$$\lim_{x \to -2} \frac{2+x}{x^2+2x}$$

2. Use the given graph to identify each limit, or state that it does not exist.



(a) $\lim_{x \to 1^-} f(x)$

(b) $\lim_{x \to 1^+} f(x)$

(c) $\lim_{x \to 1} f(x)$

(d) $\lim\limits_{x\to 2^-} f(x)$

(e) $\lim\limits_{x\to 2^+} f(x)$

(f) $\lim\limits_{x\to 2} f(x)$

(g) $\lim\limits_{x\to 3^-} f(x)$

(h) $\lim\limits_{x\to -3} f(x)$

3. Use numerical and graphical evidence to conjecture whether the limit at $x = a$ **exists. If not, describe what is happening at** $x = a$ **graphically.**

$$\lim_{x\to 1}\frac{x^2-1}{x^2-2x+1}$$

4. Evaluate the indicated limit, if it exists. Assume that $\lim\limits_{x\to 0}\frac{\sin x}{x}=1.$

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-3x+2}$$

5. Evaluate the indicated limit, if it exists. Assume that $\lim\limits_{x\to 0}\frac{\sin x}{x}=1.$

$$\lim_{x\to 0}\frac{\tan x}{x}$$

6. Evaluate the indicated limit, if it exists. Assume that $\lim\limits_{x\to 0}\frac{\sin x}{x}=1.$

$$\lim_{x\to 0}\frac{2x}{3-\sqrt{x+9}}$$

7. Evaluate the indicated limit, if it exists. Assume that $\lim_{x\to 0}\frac{\sin x}{x}=1.$

$$\lim_{x\to -1} f(x), \text{ where } f(x)=\begin{cases} x^2+1 & \text{if } x<-1 \\ 3x+1 & \text{if } x\geq -1 \end{cases}$$

8. Evaluate the indicated limit, if it exists. Assume that $\lim_{x\to 0}\frac{\sin x}{x}=1.$

$$\lim_{x\to 0}\frac{\tan x}{5x}$$

9. Use the given position function *f*(*t*) to find the velocity at time *t* = *a*.

$$f(t)=t^2+2,\ a=0$$

10. Given that $\lim_{x\to 0}\frac{\sin x}{x}=1$**, quickly evaluate** $\lim_{x\to 0}\frac{1-\cos^2 x}{x^2}$.

Written Assignment 3

Complete the following exercises based on assigned sections you learned in this module, and submit them to your mentor for correction and grading. Show all calculations.

1. Determine where *f* is continuous. If possible, extend *f* as in example 4.2 (See Section 1.4) to a new function that is continuous on a larger domain.

$$f(x)=\frac{4x}{x^2+x-2}$$

2. Determine where *f* is continuous. If possible, extend *f* as in example 4.2 (See Section 1.4) to a new function that is continuous on a larger domain.

$$f(x)=x\cot x$$

3. Determine where *f* is continuous. If possible, extend *f* as in example 4.2 (See Section 1.4) to a new function that is continuous on a larger domain.

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

4. Determine the intervals on which *f* is continuous.

$$f(x) = \sqrt{x^2 - 4}$$

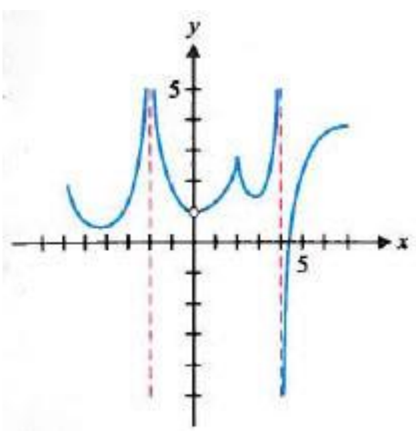
5. Determine the intervals on which *f* is continuous.

$$f(x) = (x-1)^{3/2}$$

6. Use the Intermediate Value Theorem (See Section 1.4) to verify that *f(x)* has a zero in the given interval. Then use the method of bisections to find an interval of length 1/32 that contains the zero.

$$f(x) = e^x + x, [-1, 0]$$

7. Use the given graph to identify all intervals on which the function is continuous.



8. Determine the limit (answer as appropriate, with a number, ∞, –∞, or does not exist).

$$\lim_{x \to \pi/2} x \sec^2 x$$

9. Determine the limit (answer as appropriate, with a number, ∞, –∞, or does not exist).

$$\lim_{x\to\infty}\frac{2x^2-x+1}{4x^2-3x-1}$$

10. Determine the limit (answer as appropriate, with a number, ∞, –∞, or does not exist).

$$\lim_{x\to\infty}e^{-(x+1)/(x^2+2)}$$

11. Determine all horizontal and vertical asymptotes. For each side of each vertical asymptote, determine whether $f(x)\to\infty$ or $f(x)\to-\infty$.

$$f(x)=\frac{1-x}{x^2+x-2}$$

12. Determine all horizontal and vertical asymptotes. For each side of each vertical asymptote, determine whether $f(x)\to\infty$ or $f(x)\to-\infty$.

$$f(x)=\ln(1-\cos x)$$

13. (30.) Determine all vertical and slant asymptotes.

$$y=\frac{x^2+1}{x-2}$$

14. Use graphical and numerical evidence to conjecture a value for the limit.

$$\lim_{x\to 0}\frac{\ln x^2}{x^2}$$

15. Suppose that the length of a small animal *t* days after birth is $h(t)=\dfrac{100}{2+3(0.4)^t}$ mm. What is the length of the animal at birth? What is the eventual length of the animal (i.e., the length as $t\to\infty$)?

16. Symbolically find δ in terms of ε.

$$\lim_{x \to 1} 3x = 3$$

17. Symbolically find δ in terms of ε.

$$\lim_{x \to 1} (3x + 2) = 5$$

18. Symbolically find δ in terms of ε.

$$\lim_{x \to -1} (3 - 4x) = 7$$

19. Symbolically find δ in terms of ε.

$$\lim_{x \to 1} (x^2 - x + 1) = 1$$

20. Symbolically find δ in terms of ε.

$$\lim_{x \to 0} (x^3 + 1) = 1$$

Module 3

Problems of Tangents, Velocity, Instantaneous Rates of Change

UNIT 4

Definition of Derivative

Topics

Unit 4 covers the following topics:

- developing the definition of the derivative from the slope of a secant line and using this definition to find derivatives (often called “taking the derivative the long way”)
- finding the slope of a tangent line, velocity, and the (instantaneous) rate of change

Textbook Assignment

Study sections 2.1 and 2.2 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

Finding the Derivative

You should be able to find the derivative of three types of functions using the definition of derivative. The three types of functions are:

$$f(x) = x^2$$

$$g(x) = \frac{1}{x}$$

$$h(x) = \sqrt{x}$$

Each type requires a somewhat different technique. We let $f(x)$, $g(x)$, and $h(x)$ represent each type.

First, using the definition of derivative, we have

$$\begin{aligned}\frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \frac{\Delta x(2x + \Delta x)}{\Delta x} \to 2x \text{ as } \Delta x \to 0.\end{aligned}$$

Next,

$$\begin{aligned}\frac{g(x+\Delta x) - g(x)}{\Delta x} &= \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\ &= \frac{x - (x+\Delta x)}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}\end{aligned}$$

Here we get a common denominator of

$$\frac{1}{x+\Delta x}-\frac{1}{x}$$

and instead of dividing by Δx, we multiply by $1/\Delta x$. This is equal to

$$-\frac{1}{x^2} \text{ as } \Delta \text{x} \to 0.$$

Finally,

$$\begin{aligned}\frac{h(x+\Delta x)-h(x)}{\Delta x} &= \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} \\ &= \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\cdot\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}} \\ &= \frac{x+\Delta x-x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x}} \to \frac{1}{2\sqrt{x}} \text{ as } \Delta x \to 0\end{aligned}$$

Derivative of Natural Exponential Function

You may wonder where the base of the natural logarithm e comes from. In mathematics very few numbers pop out unexpectedly. Perhaps you are most familiar with the number π of the circle. The definition of a circle is the set of all points in a plane equidistant from a given point. From that simple definition comes the astounding fact that taking the circumference of any circle and dividing by its diameter yields the same number π regardless of how large or small the circle. One would not have expected that to be a consequence of the definition of a circle.

Likewise, as a consequence of taking a derivative of the function $f(x)=\log_a x,$ e pops out unexpectedly. We show this as follows:

$$\begin{aligned}\frac{f(x+\Delta x)-f(x)}{\Delta x}&=\frac{\log_a(x+\Delta x)-\log_a x}{\Delta x}\\&=\frac{1}{\Delta x}\log_a\left(\frac{x+\Delta x}{x}\right)\\&=\frac{1}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)\\&=\frac{1}{x}\cdot\frac{x}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)\\&=\frac{1}{x}\log_a\left(1+\frac{\Delta x}{x}\right)^{x/\Delta x}\end{aligned}$$

Taking limits as $\Delta x \to 0,$ it turns out that

$$\lim_{\Delta x\to 0}\left(\frac{1+\Delta x}{x}\right)^{x/\Delta x} = e$$

for each $x \geq 0.$

You can use your calculator to estimate the limit. Thus,

$$f(x) = \frac{1}{x}\log_a e.$$

Now $\log_a e$ is a nasty number to be carrying along. But since a can be any positive base not equal to 1, and since it is simple to move from one base to another by the formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

if we let $a = e,$ we have

$$\log_c e = 1 \text{ and } f'(x) = \frac{1}{x}.$$

The number e is called the base of the natural logarithm, since e came out as a natural consequence of taking a derivative. We denote $\log_c x$ by $\ln x.$

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 2.1

Do exercises 5 and 17 on page 134 of the textbook.

Section 2.2

Do exercises 5 and 9 on page 143 of the textbook.

MODULE 3 EXERCISES

Written Assignment 4

Complete the following exercises based on assigned sections you learned in this module, and submit them to your mentor for correction and grading. Show all calculations.

1. Use Definition 1.1 (See Section 2.1) to find an equation of the tangent line to $y = f(x)$ **at** $x = a$**. Graph** $y = f(x)$ **and the tangent line to verify that you have the correct equation.**

$$f(x) = \sqrt{x+3},\ a = 1$$

2. Compute the slope of the secant line between the points at (a) $x = 1$ **and** $x = 2$, **(b)** $x = 2$ **and** $x = 3$, **(c)** $x = 1.5$ **and** $x = 2$, **(d)** $x = 2$ **and** $x = 2.5$, **(e)** $x = 1.9$ **and** $x = 2$, **(f)** $x = 2$ **and** $x = 2.1$, **and (g) use parts (a)–(f) and other calculations as needed to estimate the slope of the tangent line at** $x = 2$**.**

$$f(x) = \sqrt{x^2 + 1}$$

3. Compute the slope of the secant line between the points at (a) $x = 1$ **and** $x = 2$, **(b)** $x = 2$ **and** $x = 3$, **(c)** $x = 1.5$ **and** $x = 2$, **(d)** $x = 2$ **and** $x = 2.5$, **(e)** $x = 1.9$ **and** $x = 2$, **(f)** $x = 2$ **and** $x = 2.1$, **and (g) use parts (a)–(f) and other calculations as needed to estimate the slope of the tangent line at** $x = 2$**.**

$$f(x) = e^x$$

4. Use the position function *s* (in meters) to find the velocity at time $t = a$ **seconds.**

$$s(t) = 4/t,\ \text{(a) } a = 2;\ \text{(b) } a = 4$$

5. The function represents the position in feet of an object at time *t* seconds. Find the average velocity between (a) $t = 0$ **and** $t = 2$, **(b)** $t = 1$ **and** $t = 2$, **(c)** $t = 1.9$ **and** $t = 2$, **(d)** $t = 1.99$ **and** $t = 2$, **and (e) estimate the instantaneous velocity at** $t = 2$**.**

$s(t) = 3t^3 + t$

6. The function represents the position in feet of an object at time *t* seconds. Find the average velocity between (a) $t = 0$ **and** $t = 2$, **(b)** $t = 1$ **and** $t = 2$, **(c)** $t = 1.9$ **and** $t = 2$,
(d) $t = 1.99$ **and** $t = 2$, **and (e) estimate the instantaneous velocity at** $t = 2$**.**

$s(t) = 3\sin(t-2)$

7. (6.) Compute the derivative function f' **using Definition 2.1 or Definition 2.2 (See Section 2.2).**

$f(x) = x^2 - 2x + 1$

8. Compute the derivative function f' **using Definition 2.1 or Definition 2.2 (See Section 2.2).**

$$f(x) = \frac{2}{2x-1}$$

9. Compute the right-hand derivative $D_+ \ f(0) = \lim_{h \to 0^+} \dfrac{f(h) - f(0)}{h}$ **and the left-hand derivative** $D_- \ f(0) = \lim_{h \to 0^-} \dfrac{f(h) - f(0)}{h}$**. Does** $f'(0)$ **exist?**

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

10. Compute the right-hand derivative $D_+ \ f(0) = \lim_{h \to 0^+} \dfrac{f(h) - f(0)}{h}$ **and the left-hand derivative** $D_- \ f(0) = \lim_{h \to 0^-} \dfrac{f(h) - f(0)}{h}$**. Does** $f'(0)$ **exist?**

$$f(x) = \begin{cases} 2x & \text{if } x < 0 \\ x^2 + 2x & \text{if } x \geq 0 \end{cases}$$

Module 4

Rules for Differentiation: Product, Quotient, Chain, General Power

UNIT 5

Power, Product, and Quotient Rules and Higher-Order Derivatives

Topics

Unit 5 covers the following topics:

- using the basic shortcut rules, individually or in combinations, for taking the derivative: Constant Multiple Rule, the Sum and Difference rules, and the Power Rule
- using more complex rules, individually or in combinations, for taking the derivative: Product and Quotient rules
- computing higher order derivatives

Textbook Assignment

Study sections 2.3 and 2.4 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

Product Rule

Let $F(x) = f(x)g(x)$. The Product Rule tells us to (1) copy the first function as is, times the derivative of the second function; (2) put a plus sign in the middle and reverse the procedure; that is, (3) copy the second function as is, times the derivative of the first function. In symbols: $(fg)' = fg' + gf'$.

Quotient Rule

Let

$$F(x) = \frac{f(x)}{g(x)}.$$

The Quotient Rule tells us to (1) copy the denominator as is, times the derivative of the numerator; (2) put a minus sign in the middle of the expression and reverse the procedure; that is, (3) copy the numerator as is, times the derivative of the denominator; (4) then divide everything by the denominator squared. In symbols:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.$$

Considerations in Taking the Derivative

Among the first problems you encounter as you learn more rules for differentiating is which form of the function leads to the easiest way to take the derivative. Suppose, for example, $f(x) = (2x+1)(3x-4)$. We can either use the Product Rule

$$f'(x) = (2x+1)(3) + (3x-4)(2) = 6x + 3 + 6x - 8 = 12x - 5$$

or, just as easy, first multiply the factors

$$f(x) = (2x+1)(3x-4) = 6x^2 - 5x - 4$$

and then take the derivative term by term, $f' = 12x - 5$.

In the case $f(x) = x\sin x,$ we have no choice but to use the Product Rule, obtaining, $f'(x) = x\cos x + \sin x.$

Consider

$$f(x) = \frac{\left(x^2 - 6x + 4\right)}{\sqrt{x}}.$$

We could use the Quotient Rule to take the derivative, but the algebra involved would be formidable. Rather, it is easier to write

$$f(x) = \frac{x^2}{\sqrt{x}} - \frac{6x}{\sqrt{x}} + \frac{4}{\sqrt{x}} = x^{3/2} - 6x^{1/2} + 4x^{-1/2}.$$

Taking derivatives term by term, we get

$$f' = \frac{3}{2}x^{1/2} - 3x^{-1/2} - 2x^{-3/2}.$$

Notice that simplifying expressions algebraically does not necessarily lead to the best form when taking derivatives. An algebraic simplification usually implies no negative exponents and no radicals in the denominator. Should we choose to take a second derivative, however, the form given above is, by far, the most appropriate one to use, and

$$f' = \frac{3}{4}x^{-1/2} + \frac{3}{2}x^{-3/2} + 3x^{-5/2}.$$

Distinguishing which expressions are products and which are not is important. The function $f(x) = \sin(x+1)$ is not a product, since sin without any expression following (which represents an angle) is meaningless. The same is true for $\ln(x+1),$ since ln with no argument to follow is meaningless.

As a perceptive student, you may realize that a function such as

$$f(x) = \frac{\sin x}{x}$$

which is a quotient, can be turned into a product by writing it $f(x) = x^{-1}\sin x.$ Since we can express any quotient as a product, the temptation may be to do away with the Quotient Rule altogether. I urge you to resist this temptation and to reconsider. Applying the Quotient Rule with positive exponents results in a derivative that is almost, if not already, simplified (see example 1 below), whereas using a Product Rule with negative exponents entails having to simplify the resulting derivative (see example 2 below). You may have one less rule to learn, but you pay dearly for it.

EXAMPLE 1 (using the Quotient Rule)

$$f'(x) = \frac{(x\cos x - \sin x)}{x^2}$$

EXAMPLE 2 (using the Product Rule)

$$f'(x) = x^{-1}\cos x + \sin x(-x^{-2}) = \frac{\cos x}{x} + \frac{-\sin x}{x^2}$$
$$= \frac{x\cos x - \sin x}{x^2}.$$

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 2.3

Do exercises 7, 9, 11, 15, 21, and 27 on pages 151–152 of the textbook.

Section 2.4

Do exercises 1, 5, and 17 on pages 158–159 of the textbook.

UNIT 6

The Chain and General Power Rules

Topics

Unit 6 covers the following topics:

- using more complex rules, individually or in combinations as needed, to take derivatives: Chain and General Power rules
- applying all derivative rules individually or in combinations to polynomial, rational, trigonometric, exponential, and logarithmic functions
- interpreting phrases in applications that imply taking a derivative such as finding the slope of a tangent line, finding velocity, and finding the (instantaneous) rate of change
- computing higher order derivatives
- simplifying the derivative algebraically as much as possible (especially important when higher-order derivatives are to be taken)
- interpreting phrases in applications that imply taking a higher order derivative such as finding acceleration, which means a second derivative is needed

Textbook Assignment

Study sections 2.5, 2.6, and 2.7 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

You now possess what can be considered the arithmetic of calculus. Just as you needed to learn how to add, subtract, multiply, and divide before you were able to use arithmetic effectively, so you need to know the Product, Quotient, Chain, and General Power rules before you can effectively use differential calculus.

Chain Rule

Suppose we have two functions, $f(u) = 2u + 5$ and $u(x) = 5 - 7x$. If we replace *u* by *u*(*x*) in *f*(*u*), can we call the resulting expression *f*(*x*)? The answer is no, since by the meaning of the functional notation,

$$f(x) = 2x + 5$$

whereas

$$f(u(x)) = 2(5 - 7x) + 5 = 10 - 14x + 5 = 15 - 14x.$$

However, if we can informally agree that *f*(*x*) really means *f*(*u*(*x*)) whenever used in conjunction with the function *u*(*x*), then the Chain Rule can be stated simply as:

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}.$$

This formula has great appeal, since if we think of the derivatives as fractions (which they are not; rather they are the limit of difference quotients), then we can cross-cancel the *du*'s on the right-hand side and get

$$\frac{df}{dx}.$$

Thus, it is worth the abuse of functional language to allow the Chain Rule to be used so easily by the preceding formula. For example, if $f(u) = u^2 + 1$ and $u(x) = 3x - 2,$ then we can either replace u in the f rule by $3x - 2,$ obtaining

$$f(u(x)) = (3x-2)^2 + 1 = 9x^2 - 12x + 5$$

and

$$\frac{df}{dx} = 18x - 12$$

or use the Chain Rule,

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

where

$$\frac{df}{du} = 2u \text{ and } \frac{du}{dx} = 3.$$

Thus,

$$\frac{du}{dx} = 2u \cdot 3 = 6(3x-2) = 18x - 12.$$

Of course, you may ask, Why do we need the Chain Rule if we can simply replace u by $u(x)$ in $f(u)$ and differentiate as we did above? The reason is as follows.

Suppose

$$f(x) = \sqrt{x^2 + 1}$$

and we wish to differentiate this function. We cannot use the Power, Product, or Quotient rules to accomplish this. However, if we let $u(x) = x^2 + 1,$ then

$$f(u) = \sqrt{u} = u^{1/2}.$$

We can now use the Power Rule to find

$$\frac{df}{du} = \frac{1}{2} u^{-1/2}.$$

Likewise,

$$\frac{du}{dx} = 2x + 1$$

and by the Chain Rule,

$$\frac{df}{dx} = \frac{1}{2}u^{-1/2} \cdot 2x = \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x.$$

This application of the Chain Rule is called the General Power Rule.

General Power Rule

We can state the General Power Rule as follows: If the function is a power of some expression, you apply the Power Rule to the entire expression and then multiply by the derivative of the expression. In symbols, if $F(x) = (f(x))^n$, then

$$\frac{dF}{dx} = n(f(x))^{n-1} \cdot \frac{df}{dx}.$$

Simplifying Derivatives

Expressions like

$$f(x) = x\sqrt{2x+1}$$

not only require using a combination of the Product and General Power rules to differentiate but then must be simplified, which is a job in itself. Fortunately, a neat little trick makes simplification rather easy.

Since $f(x)$ is primarily a product, we must start with the Product Rule. We copy the first function x and multiply by the derivative of the second function, $\sqrt{2x+1}$. Since this function is an expression raised to the 1/2 power, we must use the General Power Rule. This gives us

$$\frac{1}{2}(2x+1)^{-1/2} \cdot 2 .$$

Reversing the procedure, we copy $\sqrt{2x+1}$ and take the derivative of x, which is 1. Thus,

$$\frac{dF}{dx} = x \cdot (2x+1)^{-1/2} \cdot 2 + (2x+1)^{1/2} \cdot 1.$$

The expression $(2x+1)$ is raised to two different powers that differ by one. Therefore, if we factor out the expression raised to the smaller of the powers, the larger power will become the first power. Thus,

$$(2x+1)^{-1/2}(x+(2x+1)) = \frac{3x+1}{\sqrt{2x+1}}$$

and we have simplified the derivative.

In the preceding example, we had to know that the function was primarily a product so that we could begin with the Product Rule. We must do the same thing with all types of functions. The function

$$f(x)=\sqrt{\frac{x-1}{x+1}}$$

is primarily a power the way it is written, so we must start off with the General Power Rule. However, if we write the function as

$$f(x)=\frac{\sqrt{x-1}}{\sqrt{x+1}}$$

then it is primarily a quotient, and we must start with the Quotient Rule.

In finding a derivative at a particular value, we do not need to simplify algebraically. Once we take the derivative in unsimplified form, we can replace the variable by the particular value and simplify arithmetically, which is much easier to do.

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual.* Do **not** send your work to the mentor.

Section 2.5

Do exercises 7, 11, and 27 on pages 165–166 of the textbook.

Section 2.6

Do exercises 9, 11, and 13 on page 173 of the textbook.

Section 2.7

Do exercises 17 and 21 on page 181 of the textbook.

MODULE 4 EXERCISES

Written Assignment 5

Complete the following exercises based on assigned sections you learned in this module, and submit them to your mentor for correction and grading. Show all calculations.

1. Differentiate the function.

$$h(x) = 12x - x^2 - \frac{3}{\sqrt[3]{x^2}}$$

2. Differentiate the function.

$$f(t) = 3t^{\pi} - 2t^{1.3}$$

3. Differentiate the function.

$$f(x) = \frac{4x^2 - x + 3}{\sqrt{x}}$$

4. Compute the indicated derivative.

$$f'''(t) \text{ for } f(t) = 4t^2 - 12 + \frac{4}{t^2}$$

5. Use the given position function to find the velocity and acceleration functions.

$$s(t) = -4.9t^2 + 12t - 3$$

6. Find an equation of the tangent line to $y = f(x)$ **at** $x = a$**.**

$$f(x) = x^2 - 2x + 1, \; a = 2$$

7. Find the derivative.

$$f(x) = (x^{3/2} - 4x)\left(x^4 - \frac{3}{x^2} + 2\right)$$

8. Find the derivative.

$$f(x) = \frac{x^2 - 2x}{x^2 + 5x}$$

9. Find an equation of the tangent line to the graph of $y = f(x)$ **at** $x = a$**.**

$$f(x) = (x^3 + x + 1)(3x^2 + 2x - 1), \; a = 1$$

10. Differentiate each function.

(a) $f(t) = (t^4 + 2)\sqrt{t^2 + 1}$

(b) $f(t) = \sqrt{t}(t^{4/3} + 3)$

11. Differentiate each function.

(a) $h(w) = \dfrac{(w^3 + 4)^5}{8}$

(b) $h(w) = \dfrac{8}{(w^3 + 4)^5}$

12. Find an equation of the tangent line to the graph of $y = f(x)$ **at** $x = a$**.**

$$f(x) = \frac{6}{x^2 + 4}, \; a = -2$$

13. Find the derivative.

$$f(t)=\sqrt{\cos 5t \sec 5t}$$

14. Find the derivative.

$$f(w)=w^2 \sec^2 3w$$

15. Find the derivative.

$$f(x)=4\sin^2 3x+4\cos^2 3x$$

16. Differentiate each function.

(a) $f(x)=\dfrac{\sqrt{\ln x}}{x}$

(b) $g(t)=\dfrac{\ln\sqrt{t}}{t}$

17. Differentiate each function.

(a) $f(x)=\sqrt[3]{e^{2x}x^3}$

(b) $f(w)=\sqrt[3]{e^{2w}+w^3}$

18. Find an equation of the tangent line to $y=f(x)$ at $x=1$.

$$f(x)=2\ln x^3$$

19. The value of an investment at time t is given by $v(t)$. Find the instantaneous percentage rate of change.

$$v(t)=60e^{-0.2t}$$

20. A bacterial population starts at 500 and doubles every four days. Find a formula for the population after t days and find the percentage rate of change in population.

Module 5

Implicit Differentiation and Related Rates

UNIT 7

Implicit Differentiation

Topics

Unit 7 covers the following topics:

- distinguishing between explicit and implicit functions
- differentiating implicitly and solving for the derivative
- working with implicit differentiation applications like determining the equation of the tangent line at a given point on the graph implicitly and finding the second derivative differentiating implicitly

Textbook Assignment

Study section 2.8 (pp. 183–187) in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

We have to use our imaginations when performing implicit differentiation. Let's suppose we have the implicit equation $x^2 + xy + y^2 = 4$. Let's suppose further that we want

$$\frac{dy}{dx}.$$

This means that we are treating x as an independent variable and are imagining that y is a function of x. Even though we do not know what the function is, we can think of it as an expression in x and use the appropriate rules for differentiating. The procedure is to differentiate both sides of the equation term by term. The derivative of x^2 is $2x$. However, the derivative of xy is more complicated. Since we are imagining y as some expression in x, xy is a product, and we must use the Product Rule in taking a derivative.

Thus,

$$\frac{d(xy)}{dx} = x\frac{dy}{dx} + y(1).$$

The third term, y^2, requires the General Power Rule. Hence,

$$\frac{d(y^2)}{dx} = 2y\frac{dy}{dx}.$$

Going to the right-hand side of the equation, we have

$$\frac{d(4)}{dx} = 0.$$

Thus,

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0.$$

We now solve algebraically for *dydx*. We keep every term involving a *dydx* on one side of the equation and move all other terms to the other side. We obtain

$$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - y.$$

Next, we factor out dy/dx, obtaining

$$\frac{dy}{dx}(x+2y) = -2x - y \text{ and } \frac{dy}{dx} = \frac{(-2x-y)}{(x+2y)}.$$

If we are given a point at which to evaluate *dy/dx*, then simplifying algebraically is not necessary. Once we have taken the derivative in unsimplified form, we can then replace *x* and *y* with numbers and solve arithmetically.

Let's suppose we have the same implicit equation as before, but now we want to evaluate *dy/dx* at the point $(-2,2)$. From the step where we differentiated and obtained

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

we replace *x* by –2 and *y* by 2 and get

$$-4 - 2\frac{dy}{dx} + 2 + 4\frac{dy}{dx} = 0.$$

Thus,

$$2\frac{dy}{dx} = 2 \text{ and } \frac{dy}{dx} = 1.$$

Stretching our imaginations a bit further, let's suppose we have the same implicit equation but now both *x* and *y* are functions of some other variable *t*, and we want to find *dy/dt*. Now

$$\frac{d(x^2)}{dt} = 2x\frac{dx}{dt}.$$

$$\frac{d(xy)}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} \text{ and } \frac{d(y^2)}{dt} = 2y\frac{dy}{dt}.$$

We still have

$$\frac{d(4)}{dt} = 0$$

and we obtain

$$2x\frac{dx}{dt} + x\frac{dy}{dt} + y\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

We then solve algebraically for *dy/dt*.

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 2.8

Do exercises 9 and 19 on page 191 of the textbook.

UNIT 8

Related Rates

Topics

Unit 8 covers the following topics:

- setting up and solving word problems involving time rates of change of all kinds of physical entities at a particular instant
- seeing how the rates of change of various parameters are related to each other

Textbook Assignment

Study section 3.8 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

Reading through the word problems in this section may be so intimidating that our first reaction is to believe the problems can't be solved. So, the first thing to do is not read through the entire problem right away. For success in solving related rates problems, we must first look for a formula that relates the variables involved. The formula might be the area or volume of some geometrical region, or it could be the Pythagorean theorem, etc.

In each of the problems, numbers are given along with their units. We must match these numbers with the corresponding symbol from the formula. The units make this matching relatively easy if you know the difference between length, area, and volume. For example, $4m^2$ represents the area A, whereas $4m^2$/sec represents dA/dt. Likewise, volume is given in cubic units, and the derivative of volume is given in cubic units per unit time.

We assume that all the variables are functions of time, and in many problems we are given connections between some of the variables that allow us to express one variable in terms of another simplifying the expression.

Finally, we are required to differentiate the expression implicitly with respect to time. Once we have done this, we substitute the given numbers and solve for the remaining symbol.

The volume of a cone, which is used in a number of problems, is given by

$$V = \frac{1}{3}\pi r^2 h.$$

If not given information connecting r and h, then we must use the Product Rule to differentiate. However, if we are given that $r = 3h$, say, then we can replace r in the expression and get

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3h)^2 h = \frac{1}{3}\pi(9h^2)h = 3\pi h^3.$$

This can now be easily differentiated

$$\frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt}.$$

The section includes many problems involving right triangles, where the relation between the variables comes from the Pythagorean theorem, namely, $s^2 = x^2 + y^2$. We usually let x be the horizontal variable, y the vertical variable, and *s* the hypotenuse or slant height. Differentiating implicitly, we get

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

Canceling out the 2s yields

$$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 3.8

Do exercises 7 and 25 on pages 283–284 of the textbook.

MODULE 5 EXERCISES

Written Assignment 6

Complete the following exercises based on assigned sections you learned in this module, and submit them to your mentor for correction and grading. Show all calculations.

1. Find the derivative $y'(x)$ implicitly.

$$3x + y^3 - \frac{4y}{x+2} = 10x^2$$

2. Find the derivative $y'(x)$ implicitly.

$$e^{x^2} y - 3\sqrt{y^2 + 2} = x^2 + 1$$

3. Find an equation of the tangent line at the given point. If you have a CAS that will graph implicit curves, sketch the curve and the tangent line.

$$x^3 y^2 = -2xy - 3 \text{ at } (-1, -3)$$

4. Find an equation of the tangent line at the given point. If you have a CAS that will graph implicit curves, sketch the curve and the tangent line.

$$x^4 = 8(x^2 - y^2) \text{ at } (2, -\sqrt{2})$$

5. Find the second derivative $y''(x)$.

$$(x + y)^2 - e^{y+1} = 3x$$

6. Suppose a forest fire spreads in a circle with radius changing at a rate of 5 feet per minute. When the radius reaches 200 feet, at what rate is the area of the burning region increasing?

7. For a small company spending \$*x* thousand per year in advertising, suppose that annual sales in thousands of dollars equal $s = 80 - 20e^{-0.04x}$. If the current advertising budget is $x = 40$ and the budget is increasing at a rate of \$1500 per year, find the rate of change of sales.

8. A camera tracks the launch of a vertically ascending spacecraft. The camera is located at ground level 2 miles from the launchpad. (a) If the spacecraft is 3 miles up and traveling at 0.2 mile per second, at what rate is the camera angle (measured from the horizontal) changing? (b) Repeat if the spacecraft is 1 mile up (assume the same velocity). Which rate is higher? Explain in commonsense terms why it is larger.

9. Suppose that you are blowing up a balloon by adding air at the rate of 1 ft^3/s. If the balloon maintains a spherical shape, the volume and radius are related by $V = \frac{4}{3}\pi r^3$. Compare the rate at which the radius is changing when *r* = 0.01 ft versus when *r* = 0.1 ft. Discuss how this matches the experience of a person blowing up a balloon.

10. Sand is dumped such that the shape of the sandpile remains a cone with height equal to twice the radius. (a) If the sand is dumped at the constant rate of 20 ft^3/s, find the rate at which the radius is increasing when the height reaches 6 feet. (b) Repeat for a sandpile for which the edge of the sandpile forms an angle of 45° with the horizontal.

Module 6

Maxima and Minima Theory

UNIT 9

Graphing Mostly Polynomials

Topics

Unit 9 covers the following topics:

- graphing a function by determining:
  - the critical values of the function
  - where the function increases and decreases
  - the absolute maximum and minimum on a closed interval
  - the relative extrema using the First Derivative Test
  - the general points of inflection
  - where the function is concave upward and where it is concave downward
  - the relative extrema using the Second Derivative Test

Textbook Assignment

Study sections 2.10, 3.3, 3.4, and 3.5 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

Ever since René Descartes connected the then separate fields of algebra and geometry into what is now known as analytic geometry, the persistent question has been how much weight to give the geometric aspects of the subject and how much the algebraic aspects. Over the centuries, creating the proper balance between the two has shifted back and forth. At present, since the coming of age of the graphing calculator, many calculus courses allow too much of geometry to take hold at the expense of algebra. This is unfortunate. Although the calculator graphs the function and can give the needed information, without the algebra and analysis, the student obtains only a superficial comprehension of this very beautiful and profound subject.

On the other hand, some authors of calculus textbooks go overboard on the algebra and analysis. The student is required first to find where the function is increasing or decreasing, then to find the critical values and test what they are, then to find where the curve is concave upward or downward, where the general points of inflection are, and finally, as a last step, to graph the function. By this time the student is so overwhelmed with the details that the subject ceases to be one where much of this information can be obtained by just looking at the graph.

In this course we shall use the least amount of analysis in order to graph a quick sketch of the curve. Then we shall use this sketch to answer such questions as where the function is increasing, where it is concave upward, what general inflection points are showing on the graph. Thus, we shall make use of the best features of both analysis and geometry to simplify the subject without making it superficial.

The steps used to graph the function are as follows:

1. Take the derivative of $f(x)$.

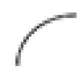
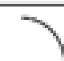
2. Set the derivative equal to zero and solve for x. (These will give you the critical values of the function.)

3. Test for extrema using either the First or Second Derivative Test.

4. Find the y-coordinates of the critical values and plot just those points on a coordinate system.

5. Connect with a smooth curve after having drawn the top of a hill at the critical point if there is a maximum and the bottom of a hill if there is a minimum or level-off at the critical point, and then continue in the same direction if it is a point of inflection. (The top of a hill at a critical point looks like $\cap$, whereas the bottom of a hill looks like $\cup$.)

6. The key element to remember is that on any interval not containing a critical point only one of four things can occur:

F′	F″	Curve shape of F is:	Picture
+	+	Increasing and concave up	
+	-	Increasing and concave down	
-	+	Decreasing and concave up	
-	-	Decreasing and concave down	

The Second Derivative Test is much easier to use than the First Derivative Test when graphing polynomials, since it is so easy to take a second derivative. However, if the second derivative at the critical point is 0, then the test fails and we have to use a First Derivative Test, which never fails.

If the first derivative of the function at $x = c$ is undefined but $f(c)$ is defined, then c is also a critical value of $f(x)$. In this case we must use a First Derivative Test, since the second derivative is also undefined at c.

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 2.10

Do exercises 3 and 43 on pages 204–205 of the textbook.

Section 3.3

Do exercises 13, 15, and 27 on page 241 of the textbook.

Section 3.4

Do exercises 11, 27, and 29 on page 249 of the textbook.

Section 3.5

Do exercises 1, 11, 13, 15, and 37 on page 257 of the textbook.

UNIT 10

Graphing Rational Functions

Topics

Unit 10 covers the following topics:

- finding the vertical and horizontal asymptotes of the function
- finding the critical values of the function
- making a quick sketch of the function using the preceding information along with the plotting of a point in each section of the plane partitioned by the vertical asymptotes

Textbook Assignment

Study section 3.6 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

To graph rational functions using a minimum of analysis, we must, in addition to the procedure for graphing polynomials described in unit 9, first find the vertical asymptotes and at most one horizontal asymptote.

Rational functions are ratios of polynomials, that is,

$$R(x) = \frac{P(x)}{Q(x)}.$$

The first step is to find the vertical asymptotes. To do this, we set the denominator equal to zero and solve for *x*. If $x = a$ is such a solution and $P(a) \neq 0,$ then $x = a$ is a vertical line that the curve approaches but never reaches. If both $Q(a) = P(a) = 0,$ then there is a hole in the curve at $(a, R(a)),$ but no vertical asymptote.

To find the horizontal asymptote, if there is one, we take

$$\lim_{x \to \infty} R(x).$$

It turns out that $\lim_{x \to \infty} R(x)$ will be the same if the limit exists and is finite. This means there is only one horizontal asymptote.

An easy way to find the limit of *R*(*x*) as $x \to 0$ is to use "The Rule":

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)} = L.$$

1. If $\deg P = \deg Q,$ then L equals the ratio of leading coefficients, where $\deg P$ means "the degree of the polynomial *P*(*x*)."

2. If $\deg P < \deg Q,$ then $L = 0.$

3. If $\deg P > \deg Q,$ there is no finite limit.

In each case where there is a limit *L*, $y = L$ is the horizontal asymptote.

EXAMPLE

To illustrate our procedure of finding vertical and horizontal asymptotes, let

$$R(x) = \frac{2x^2}{(1-x^2)}.$$

First we find the vertical asymptotes. Setting the denominator equal to zero and solving for *x*, we have

$$\begin{aligned} 1-x^2 &= 0 \\ (1-x)(1+x) &= 0 \\ x &= 1, -1 \end{aligned}$$

Neither value makes the numerator equal to zero. Therefore, the lines $x = 1$ and $x = -1$ are vertical asymptotes. These lines separate the plane into three sections—to the left of both lines, between the lines, and to the right of the lines. We must have a representative point in each section for our sketch.

We now find the horizontal asymptote of *R*(*x*). To do so we take

$$\lim_{x \to \infty} \frac{2x^2}{(1-x^2)}.$$

Since the polynomial $2x^2$ has degree 2 and the polynomial $1-x^2$ also has degree 2, the limit is the ratio of leading coefficients by "The Rule." The leading coefficient of the numerator (i.e., the coefficient of the term with the largest power of *x*) is 2. The leading coefficient of the denominator is –1. Hence, $L = 2/-1 = -2,$ and $y = -2$ is the horizontal asymptote.

The last part of the analysis is to take the derivative of *R*(*x*), set it to equal zero, find the critical values, and see what they are. Since *R*(*x*) is a quotient, we must use the Quotient Rule. Thus,

$$\frac{dR(x)}{dx} = \frac{(1-x^2)(4x) - 2x^2(-2x)}{(1-x^2)^2} = \frac{4x}{(1-x^2)^2}.$$

We must set this equal to zero and solve for *x*. An easy way to do this is to write

$$\frac{4x}{(1-x^2)^2} = \frac{0}{1}$$

and then cross-multiply, obtaining $4x = 0(1-x^2)^2,$ or $x = 0.$

If we do not require finding the general points of inflection, then it is much easier to use a First Derivative Test rather than a Second Derivative Test, since we do not have to take another derivative using the Quotient Rule. From the First Derivative Test using $x = -1/2$ and $x = 1/2$ on either side of 0, we see that at $x = 0$ we have a relative minimum. We must plot the critical points, which in this case yield a value between the two asymptotes.

We then can select any value of x less than –1, say $x = -2$, and any value greater than 1, say $x = 2$. Since the curve has to approach the asymptotes and there is only one critical value, we will be able to make a quick sketch of the curve.

Note that the curve cannot cross any vertical asymptote, since that would put a zero in the denominator, which is not allowed. It can, however, cross a horizontal asymptote in the middle region, for it is only as $x \to \infty$ or $x \to -\infty$ that the curve approaches the asymptote without ever reaching it.

If $x = c$ is a critical value because

$$\frac{dR(c)}{dx}$$

has a zero in the denominator but not in the numerator, in which case it is undefined, and $R(c)$ is defined, then the curve has a vertical tangent at the critical point and instead of graphing a minimum as in the example above, it looks rather like)(.

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 3.6

Do exercises 1, 9, 11, and 13 on page 267 of the textbook.

UNIT 11

Applied Maxima and Minima

Topics

Unit 11 covers the following topics:

- translating word problems into mathematical expressions and equations
- using maxima and minima theory to solve the problems
- identifying which problems have endpoints that must be considered along with the critical points
- applying geometric formulas for area, surface area, volume, etc., of various geometric figures

Textbook Assignment

Study section 3.7 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

Most of the word problems in section 4.7 involve an equation in two variables and an expression also in two variables. You are asked to maximize or minimize the expression. The procedure is as follows:

1. Solve the equation for one of the unknowns.
2. Substitute it into the expression. (The expression now contains only one variable.)
3. Take the derivative, set it equal to zero, and solve obtaining the critical values.
4. If necessary, test the critical values to see which one gives the sought-after extrema.

Consider the following problem:

PROBLEM

Find two numbers whose sum is 20 and whose product is as large as possible.

Solution Let x and y represent the two numbers. Then $x + y = 20.$ This is the equation. The expression is the thing we want to maximize, in this case the product $P = xy.$ We solve the equation for y obtaining $y = 20 - x$ and substitute this into the expression. Hence,

$$P = x(20 - x) = 20x - x^2.$$

Differentiating with respect to x yields

$$\frac{dP}{dx} = 20 - 2x.$$

Setting this equal to zero, we get $20 - 2x = 0,$ implying that $x = 10.$ This is our only critical value, and therefore it must be the solution or else there is no solution. However, in this case it is simple enough to take a second derivative, obtaining –2, and by the Second Derivative Test it is a maximum. Thus, the numbers are 10 and 10, and the problem is solved.

Applied maxima and minima problems are among the most widely used techniques in the physical and social sciences. Everyone in the sciences must always be concerned with problems such as the most weight something can hold, the smallest force needed to move something, the maximum permissible dose of a medicine, the maximum profit or the minimum cost for some part of a business. Once a mathematical model can be set up for any of these situations, we can then use the theory to solve the problems.

Many problems can be simplified by using the square of the expression instead of the expression itself. For example, suppose we have the following problem:

PROBLEM

Find the rectangle of largest area that can be inscribed in a semicircle of radius 4.

Solution Let x represent the horizontal distance from the center of the semicircle to the vertex on the right of the rectangle. Let y represent the height of the rectangle. Then the area of the rectangle is $A = 2xy$. This is the expression we wish to maximize. The equation connecting x and y is the Pythagorean theorem given by $x^2 + y^2 = 16$. Solving the equation for y yields

$$y = \sqrt{16 - x^2}$$

and

$$A = 2x\sqrt{16 - x^2}\ .$$

This is a rather nasty derivative to take and is unnecessary. If we let the expression be $A^2 = 4x^2y^2$, then we can solve the equation for y^2 instead of y and $A^2 = 4x^2(16 - x^2)$, which is a simple derivative to take and leads to the same answer, since the maximum area leads to the same rectangle as the maximum squared area.

Occasionally we have a word problem that has endpoints, and we must check these as well to find the absolute extreme. For example, consider the following problem:

PROBLEM

A 20-inch piece of wire is to be cut into two parts, one part forming a square and the other a circle. Find the dimensions of the square and circle that minimize and maximize the sum of the areas.

Solution Besides finding the equation and expression and then the critical values, we must also consider the case where the entire wire is used to form a rectangle and the area of the circle is 0, and the case where the entire wire is used to form the circle and the area of the square is 0. These possibilities must be considered along with the critical values and represent the endpoints of the problem.

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 3.7

Do exercises 7, 17, and 19 on pages 276–277 of the textbook.

MODULE 6 EXERCISES

Written Assignment 7

Complete the following exercises based on assigned sections you learned in this module, and submit them to your mentor for correction and grading. Show all calculations.

1. Check the hypotheses of Rolle's Theorem and the Mean Value Theorem (See Section 2.10), and find a value of *c* that makes the appropriate conclusion true.

$$f(x) = x^3 + x^2, [-1,1]$$

2. Explain why it is not valid to use the Mean Value Theorem (See Section 2.10). When the hypotheses are not true, the theorem does not tell you anything about the truth of the conclusion. Find the value of *c*, or show that there is no value of *c* that makes the conclusion of the theorem true.

$$f(x) = x^{1/3}, [-1,1]$$

3. Find all critical numbers by hand. If available, use graphing technology to determine whether the critical number represents a local maximum, local minimum, or neither.

$$f(x) = \frac{x^2 - x + 4}{x - 1}$$

4. Find the absolute extrema of the given function on each indicated interval.

$f(x) = x^4 - 8x^2 + 2$ on (a) [–3, 1] and (b) [–1, 3]

5. Find the absolute extrema of the given function on each indicated interval.

$f(x) = \sin x + \cos x$ on (a) [0, 2π] and (b) [π/2, π]

6. Find (by hand) all critical numbers and use the First Derivative Test (See Section 3.4) to classify each as the location of a local maximum, local minimum, or neither.

$$y = x^5 - 5x^2 + 1$$

7. Find (by hand) all critical numbers and use the First Derivative Test to classify each as the location of a local maximum, local minimum, or neither.

$$y = x^2 e^{-x}$$

8. Find (by hand) all critical numbers and use the First Derivative Test to classify each as the location of a local maximum, local minimum, or neither.

$$y = \frac{x}{1+x^4}$$

9. Determine the intervals where the graph of the function is concave up and concave down, and identify inflection points.

$$f(x) = x + 3(1-x)^{1/3}$$

10. Find all critical numbers and use the Second Derivative Test (See Section 3.5) to determine all local extrema.

$$f(x) = e^{-x^2}$$

11. Find all critical numbers and use the Second Derivative Test (See Section 3.5) to determine all local extrema.

$$f(x) = \frac{x^2 - 1}{x}$$

12. Determine all significant features by hand, and sketch a graph.

$$f(x) = x \ln x$$

13. Determine all significant features by hand, and sketch a graph.

$$f(x) = x^{2/3} - 4x^{1/3}$$

14. Graph the function, and completely discuss the graph as in example 6.2 (See Section 3.6) of the text.

$$f(x) = x^4 + 4x^3 - 1$$

15. Graph the function, and completely discuss the graph as in example 6.2 (See Section 3.6) of the text.

$$f(x) = \frac{x-4}{x^3}$$

16. Graph the function, and completely discuss the graph as in example 6.2 (See Section 3.6) of the text.

$$f(x) = \sqrt{x^3 - 3x^2 + 2x}$$

17. Graph the function, and completely discuss the graph as in example 6.2 (See Section 3.6) of the text.

$$f(x) = x^3 - \frac{3}{400}x$$

18. A box with no top is to be built by taking a 12″-by-16″ sheet of cardboard, cutting *x*-in. squares out of each corner and folding up the sides. Find the value of *x* that maximizes the volume of the box.

19. Following example 7.5 (See Section 3.7) in the text, we mentioned that real soda cans have a radius of about 1.156″. Show that this radius minimizes the cost if the top and bottom are 2.23 times as thick as the sides.

20. A company needs to run an oil pipeline from an oil rig 25 miles out to sea to a storage tank that is 5 miles inland. The shoreline runs east-west and the tank is 8 miles east of the rig. Assume it costs $50 thousand per mile to construct the pipeline under water and $20 thousand per mile to construct the pipeline on land. The pipeline will be built in a straight line from the rig to a selected point on the shoreline, then in a straight line to the storage tank. What point on the shoreline should be selected to minimize the total cost of the pipeline?

***Note:** Submitting a graph is not required; however, you are encouraged to create one for your own benefit. In your solution, simply list all relative information for the specific problem such as complete points of the form (x, y) for any local extrema, inflection points, and intercepts. If needed, give equations for asymptotes, and list any discontinuities.

Module 7

Antiderivatives and the Indefinite and Definite Integral

UNIT 12

Antiderivatives

Topics

Unit 12 covers the following topics:

- the basic concept of an antiderivative
- picturing geometrically what the curves look like for different values of the arbitrary constant
- applying the basic integration rules
- solving a differential equation for a particular solution using the indefinite integral and initial conditions
- applying the concepts learned to problems in science

Textbook Assignment

Study section 4.1 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

Integration

Unlike differentiation, integration does not have a Product Rule or Quotient Rule. The only rule at our disposal for integration is the Power Rule, when we are not dealing with a function that is the known derivative of another function.

In succeeding sections you will learn different methods of integration in which we change variables in such a way that we can apply the Power Rule for integration and then return to our original variable. The Power Rule for integration is as follows:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1 \text{ and } \int \frac{1}{x} dx = \ln|x| + C.$$

Initial Condition

If we are trying to determine the path of a particle and we integrate with the indefinite integral, we will only know the function representing the path up to an arbitrary constant. Since that constant can be any value, however, the path of the particle can be anywhere. Only by knowing some initial condition can we evaluate the constant to obtain a unique function that represents the path of the particle. And only then can scientists use the information.

If x is raised to a fractional power, then using the Power Rule to integrate, we add 1 to the exponent and divide by the new power. However, dividing by a fraction is the same as multiplying by its reciprocal. Thus, for example,

$$\int x^{2/3} dx = \frac{3}{5} x^{5/3} + C\,.$$

In taking derivatives, we may think of d/dx as an operator acting on some expression in x, resulting in its derivative. Likewise, in integrating, we may think of

$$\int (\ )dx$$

as an operator, operating on some expression in x that is put into the parentheses and yields the collection of antiderivatives of the expression.

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 4.1

Do exercises 15, 17, 21, and 45 on pages 307–308 of the textbook.

UNIT 13

The Definite Integral

Topics

Unit 13 covers the following topics:

- using the sigma notation Σ
- approximating the area of a region in the plane using upper and lower sums
- using the limit of these sums to find the area of the region
- generalizing to Reimann sums and defining the definite integral using Reimann sums
- evaluating the definite integral of simple functions using the definition of definite integral

Textbook Assignment

Study sections 4.2, 4.3, and 4.4 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

In the definition of a derivative, we first found two points (x, $f(x)$) and ($x + \Delta x$, $f(x + \Delta x)$) on the curve and took the slope of the secant line joining them,

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

as an approximation to the slope of the tangent line. The final step was to take a limit as one point approached the other (as $\Delta x \to 0$). This gave us the actual slope of the tangent line at (x, $f(x)$).

Similarly, suppose $y = f(x)$ is a function whose graph lies above the x-axis in the interval a, b. To find the area of the region bounded by the curve, the x-axis, and the vertical lines $x = a$ and $x = b$, we first approximate the area using rectangles.

To do this we first divide the interval with points of subdivision $x_0, x_1, x_2, \cdots x_n$, where $x_0 = a < x_1 < x_2 < \cdots < x_n = b$. For the most general definition, we do not assume that these points of subdivision are evenly spaced. We consider the first interval x_0, x_1 and select any point c_1 in that interval. Then the height of the approximating rectangle will be $f(c_1)$ and the base will be $x_1 - x_0 \equiv \Delta x_1$. Thus the area of the rectangle is $f(c_1)\Delta x$. Moving on to the second rectangle x_2, x_1, we select an arbitrary point c_2 in that interval and form the rectangle whose area is $f(c_2)\Delta x_2$. Doing this in each of the intervals and adding the areas of each rectangle formed yields

$$f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n.$$

This gives us an approximation to the desired area. Using summation notation, this can be written

$$\sum_{i=1}^{n} f(c_i)\Delta x_i.$$

This is the **Reimann sum.**

Notice that for derivatives, the difference quotient, which represents the slope of the secant line, can be written as $\Delta y / \Delta x$. This expression approximates the slope of the tangent line. Upon taking limits as $\Delta x \to 0,$ we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

where we have changed from the Greek letter delta to our own letter *d*. We continue this convention with integral notation. To get better and better approximations for the area, we need narrower and narrower rectangles.

To accomplish this, we must keep adding more and more points of subdivision. Since there are $n+1$ points of subdivision, we want $n \to \infty$. Otherwise, on those intervals that do not shrink to zero, the approximation will not get better and better. Thus, the limit called for is

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i .$$

This number represents the area of the region. Greek letters Σ and Δ, which represent a capital *S* and lowercase *d*, respectively, change to our own alphabet ∫ and *d* and becomes

$$\int_a^b f(x)dx.$$

This is called the **definite integral.**

The definite integral represents the area under the curve above the *x*-axis if the curve lies above the *x*-axis. However, the definite integral is defined for all functions regardless of where the curve is. If completely below the *x*-axis over the interval, the definite integral represents the negative of the area. If partly above and partly below the *x*-axis, it represents the area of the part above minus the area of the part below.

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 4.2

Do exercises 5 and 13 on page 315 of the textbook.

Section 4.3

Do exercises 5, 13, and 15 on page 321 of the textbook.

Section 4.4

Do exercises 13, 15, 21, and 29 on page 332 of the textbook.

MODULE 7 EXERCISES

Written Assignment 8

Complete the following exercises based on assigned sections you learned in this module, and submit them to your mentor for correction and grading. Show all calculations.

1. Find the general antiderivative.

$$\int 4\frac{\cos x}{\sin^2 x}\,dx$$

2. Find the general antiderivative.

$$\int (4x - 2e^x)\,dx$$

3. Find the general antiderivative.

$$\int \frac{3}{4x^2+4}\,dx$$

4. Find the function $f(x)$ satisfying the given conditions.

$$f''(x) = 20x^3 + 2e^{2x},\; f'(0) = -3,\; f(0) = 2$$

5. Determine the position function if the velocity function is $v(t) = 3e^{-t} - 2$ and the initial position is $s(0) = 0$.

6. Write out all terms and compute the sums.

$$\sum_{i=3}^{7} (i^2 + i)$$

7. Use summation rules to compute the sum.

$$\sum_{n=1}^{140}(n^2+2n-4)$$

8. Use summation rules to compute the sum.

$$\sum_{i=4}^{20}(i-3)(i+3)$$

9. Compute sum of the form $\sum_{i=1}^{n} f(x_i)\Delta x$ **for the given values of** x_i **.**

$$f(x)=3x+5;\ x=0.4,\ 0.8, 1.2, 1.6,\ 2.0;\ \Delta x=0.4;\ n=5$$

10. Compute sum of the form $\sum_{i=1}^{n} f(x_i)\Delta x$ **for the given values of** x_i **.**

$$f(x)=x^3+4;\ x=2.05,\ 2.15,\ 2.25,\ 2.35, \ldots, 2.95;\ \Delta x=0.1;\ n=10$$

11. Approximate the area under the curve on the given interval using *n* rectangles and the evaluation rules (a) left endpoint, (b) midpoint, and (c) right endpoint.

$y=x^2+1$ on [0, 2], *n* = 16

12. Approximate the area under the curve on the given interval using *n* rectangles and the evaluation rules (a) left endpoint, (b) midpoint, and (c) right endpoint.

$y=e^{-2x}$ on [–1, 1], *n* = 16

13. Use Riemann sums (See Section 4.3) and a limit to compute the exact area under the curve.

$y=x^2+3x$ on (a) [0, 1]; (b) [0, 2]; (c) [1, 3]

14. Use Riemann sums (See Section 4.3) and a limit to compute the exact area under the curve.

$y=4x^2-x$ on (a) [0, 1]; (b) [–1, 1]; (c) [1, 3]

15. Construct a table of Riemann sums as in example 3.4 (See Section 4.3) of the text to show that sums with right-endpoint, midpoint, and left-endpoint evaluation all converge to the same value as $n \to \infty$.

$f(x) = \sin x, [0, \pi / 2]$

16. Evaluate the integral by computing the limit of Riemann sums.

$\int_{-2}^{2} (x^2 - 1) dx$

17. Write the given (total) area as an integral or sum of integrals.

The area above the *x*-axis and below $y = 4x - x^2$

18. Use the given velocity function and initial position to estimate the final position *s*(*b*).

$v(t) = 30e^{-t/4}, \quad s(0) = -1, \quad b = 4$

19. Compute the average value of the function on the given interval.

$f(x) = x^2 + 2x, [0,1]$

20. Use the Integral Mean Value Theorem (See Section 4.4) to estimate the value of the integral.

$\int_{-1}^{1} \frac{3}{x^3 + 2} dx$

Module 8

Integration Techniques and Logarithmic and Exponential Functions

UNIT 14

Fundamental Theorem of Calculus; Integration by Substitution

Topics

Unit 14 covers the following topics:

- using the Fundamental Theorem of Calculus to find the definite integral of a variety of functions, evaluate the integral of the absolute value of a function, and find area
- calculating the average value of a function over an interval
- using the definite integral to define a function
- applying the Second Fundamental Theorem of Calculus

Textbook Assignment

Study sections 4.5 and 4.6 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

Fundamental Theorem of Calculus

Only after you have evaluated the definite integral using the definition can you appreciate the Fundamental Theorem of Calculus. To evaluate

$$\int_a^b f(x)dx$$

it is only necessary to find a single antiderivative $F(x)$ of $f(x)$ and to evaluate $F(b) - F(a)$. This number gives us the answer.

The easiest way to find an antiderivative is to take the indefinite integral and set the arbitrary constant C equal to zero. If we set it equal to any other value, we would still get the same answer, since when we subtract the antiderivative at the upper and lower limits, the constant value drops out.

We can derive many properties of the definite integral quite easily from the Fundamental Theorem:

$$\int_a^a f(x)dx = F(a) - F(a) = 0$$

$$\int_b^a f(x)dx = F(a) - F(b) = -(F(b) - F(a)) = -\int_a^b f(x)dx$$

Assume $f(x)$ is integrable everywhere, and let c be any real number. Then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

This is true because the first integral on the right side of the equation equals $F(c) - F(a)$ and the second integral equals $F(b) - F(c)$. Adding them together, we get $F(b) - F(a)$.

Integration by Substitution

The first method of integration you learn in this unit is that of substitution. The idea behind this method is to change variables in such a way that the resulting integral in the new variable is just a power. Where we can then use the Power Rule, we also make use of the property that

$$\int cf(x)dx = c\int f(x)dx$$

if *c* is a constant. That is, constants can move in and out of the integrand at will without changing the expression. This is true only for constants. We cannot move an expression involving a variable outside the integrand. Consider the following example:

EXAMPLE

$$\int x\sqrt{x^2+1}\,dx$$

We usually let *u* be the expression that is raised to a power. In this case, let $u = x^2 + 1$. Then $du = 2x\,dx$. In the integrand we see an $x\,dx$, but we are missing a 2. If we had $2x\,dx$ in the integrand, we could replace it by *du*. Fortunately, we can use the property concerning constants and write

$$\int \frac{1}{2}\sqrt{x^2+1}\,2x\,dx.$$

Notice that all we have done is multiply the integrand by 1, which does not change the expression. However, now we can take the 1/2 outside the integrand, and, replacing $x^2 + 1$ by *u* and $2x\,dx$ by *du*, we get

$$\frac{1}{2}\int u^{1/2}du.$$

This is just a power of *u* and can be integrated using the Power Rule. We obtain

$$\frac{1}{2}\cdot\frac{2}{3}\cdot u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C.$$

The substitution method can only be used under the proper circumstances. Had we tried to use it with the integral

$$\int x\sqrt{x^3+1}\,dx$$

we would let $u = x^3 + 1.$ Then $du = 3x^2\ dx.$ However, we only have an $x\,dx$ in the integrand and not an $x^2 dx.$ We can easily handle the 3 in the expression for *du* but not the x^2. We cannot integrate this integral using the substitution method.

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 4.5

Do exercises 9, 13, 35, and 39 on page 341 of the textbook.

Section 4.6

Do exercises 11, 15, 31, 35, and 39 on pages 350 of the textbook.

UNIT 15

Logarithmic and Exponential Functions

Topics

Unit 15 covers the following topics:

- using long division of polynomials to change the polynomial equivalent of an improper fraction into the equivalent of a mixed number
- using the substitution method in a more creative way when necessary
- using trigonometric identities to change the expression so that integration is possible
- using logarithmic differentiation when needed
- applying the concepts learned to problems in the physical and social sciences

Textbook Assignment

Study section 4.8 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

The equivalent of an improper fraction with respect to rational functions is when the degree of the polynomial in the numerator is greater than or equal to that in the denominator. For example, let

$$f(x)=\frac{x^2+3x+5}{x-3}.$$

We can use long division or synthetic division to change the expression. Thus, we obtain

$$x+6+\frac{23}{x-3}.$$

This expression can be integrated term by term yielding

$$\frac{1}{2}x^2+6x+23\ln|x-3|+C.$$

Creative Use of Substitution

We can use the substitution method more creatively. Suppose we have

$$\int x\sqrt{x+1}\,dx.$$

If we let $u=x+1,$ then $du=dx.$ But we still have an x in the integrand for which we must account. However, since $u=x+1,$ we can solve for x in terms of u obtaining $x=u-1.$ Substitution into the integrand yields

$$\begin{aligned}\int(u-1)\sqrt{u}\,du=\int(u^{3/2}-u^{1/2})\,du&=\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}+C\\&=\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+C.\end{aligned}$$

Trigonometric Identities

The more trigonometric identities you know, the easier it will be to recognize the potential for changing an expression that at first glance cannot be integrated. For example, suppose we have

$$\int \tan x \, dx.$$

We do not recognize $\tan x$ as the derivative of any known function. However, if we remember that

$$\tan x = \frac{\sin x}{\cos x}$$

then we can express the integral as

$$\int \frac{\sin x}{\cos x} dx$$

and use substitution. Letting $u = \cos x$, then $du = -\sin x \, dx$, and we have

$$-\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C.$$

As another example, consider the integral

$$\int (1 + \tan^2 x) \, dx.$$

If we remember that the expression in the integrand is equal to $\sec^2 x$, then the integral is trivial, since $\sec^2 x$ is the derivative of $\tan x$, and the indefinite integral is just $\tan x + C$.

Logarithmic Differentiation

Logarithmic differentiation must be used when both the base and power are functions of the variable. Let $f(x) = x^x$. If it were e^x, then the derivative would also be e^x. If it were x^e, then the Power Rule would apply and the derivative would be ex^{e-1}. But here both the base and exponent are functions of *x*, and what we do is to first let $y = x^x$ and take logarithms of both sides. Thus, $\ln y = \ln x^x = x \ln x$ by the rule for logarithms. We now differentiate implicitly and get

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln x = 1 + \ln x.$$

Solving for *dy/dx*, we get

$$\frac{dy}{dx} = y(1 + \ln x) = x^x(1 + \ln x).$$

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 4.8

Do exercises 11, 15, and 21 on page 372 of the textbook.

MODULE 8 EXERCISES

Written Assignment 9

Complete the following exercises based on assigned sections you learned in this module, and submit them to your mentor for correction and grading. Show all calculations.

1. Use Part I of the Fundamental Theorem (See Section 4.5) to compute the integral exactly.

$$\int_{\pi/4}^{\pi/2} 3\csc x \cot x \, dx$$

2. Use Part I of the Fundamental Theorem (See Section 4.5) to compute the integral exactly.

$$\int_{-1}^{1} \frac{4}{1+x^2} \, dx$$

3. Use Part I of the Fundamental Theorem (See Section 4.5) to compute the integral exactly.

$$\int_{0}^{t} (\sin^2 x + \cos^2 x) \, dx$$

4. Find the position function *s*(*t*) from the given velocity or acceleration function and initial value(s). Assume that units are feet and seconds.

$$a(t) = 16 - t^2, \quad v(0) = 0, \quad s(0) = 30$$

5. Find an equation of the tangent line at the given value of *x*.

$$y = \int_{-1}^{x} \ln(t^2 + 2t + 2) \, dt, \; x = -1$$

6. Find the average value of the function on the given interval.

$$f(x) = 2x - 2x^2, [0,1]$$

7. Find the average value of the function on the given interval.

$$f(x) = e^x, [0, 2]$$

8. Evaluate the integral.

$$\int e^x \sqrt{e^x + 4}\, dx$$

9. Evaluate the integral.

$$\int \sec^2 x \sqrt{\tan x}\, dx$$

10. Evaluate the integral.

$$\int x^2 \sec^2 x^3\, dx$$

11. Evaluate the definite integral.

$$\int_1^3 x \sin(\pi x^2)\, dx$$

12. Evaluate the definite integral.

$$\int_0^2 t^2 e^{t^3}\, dt$$

13. Evaluate the definite integral.

$$\int_0^2 \frac{e^x}{1 + e^x}\, dx$$

14. Evaluate the definite integral.

$$\int_0^1 \frac{x}{\sqrt{x^2 + 1}}\, dx$$

15. Use the properties of logarithms to rewrite the expression as a single term.

$$2\ln\left(\tfrac{1}{3}\right) - \ln 3 + \ln\left(\tfrac{1}{9}\right)$$

16. Evaluate the derivative using properties of logarithms where needed.

$$\frac{d}{dx}[\ln\,(x^5 \sin x \cos x)]$$

17. Evaluate the derivative using properties of logarithms where needed.

$$\frac{d}{dx}\left(\ln\sqrt{\frac{x^3}{x^5+1}}\right)$$

18. Evaluate the integral.

$$\int \frac{1}{\sqrt{1-x^2}\,\sin^{-1} x}\,dx$$

19. Evaluate the integral.

$$\int \frac{\sin(\ln x^3)}{x}\,dx$$

20. Evaluate the integral.

$$\int_1^2 \frac{\ln x}{x}\,dx$$

Module 9

Area between Curves

UNIT 16

Area between Curves

Topics

Unit 16 covers the following topics:

- sketching and shading the enclosed region whose area you are asked to calculate
- finding the points of intersection of two curves that enclose a region
- setting up the definite integral when horizontal representative rectangles are called for
- handling regions where the curves intersect in more than two points
- applying the skills learned to problems in the physical and social sciences

Textbook Assignment

Study section 5.1 in the textbook. Then read the Technical Commentary below for further explanations and notes on the material covered.

Following the commentary are self-check exercises. Be sure to work through these practice exercises to help solidify the skills learned in the unit and to prepare for the module-ending written assignment.

Technical Commentary

In graphing the enclosed region, remember that the coordinate axes play no role except when they are given as one of the curves.

Without knowing the limits of integration, you won't be able to calculate the area. Sometimes the limits are given. Sometimes they will be obvious from the graph. However, if given two curves that enclose a region, you must solve the equations simultaneously to find the limits.

A good idea is to graph a representative rectangle and make certain that wherever it is drawn in the region, it always has the same curve on top and the same curve on the bottom.

If that does not happen, then you must split the region into two parts and calculate the definite integral corresponding to each part.

When the curves are given in the form $x = f(y)$, we can keep that form and use horizontal instead of vertical rectangles to approximate the area and set up the definite integral as a function of y instead of x. Of course in this case, we must find the limits of integration with respect to y rather than x.

Let's suppose that two curves define the enclosed region, but when we solve simultaneously to find the limits of integration, we find three points of intersection. This may mean that the curve on top has crossed over somewhere in the region and is now on the bottom. Or it may mean that the upper curve just touches the lower one and remains on the top. In the first case, we must use two different integrals, since the curves change position. In the second case, one integral will suffice, since there is no change of position and the point of intersection in the middle can be ignored.

Practice Exercises

Work through the following practice exercises from the textbook. Then check your solutions with those in the *Student Solutions Manual*. Do **not** send your work to the mentor.

Section 5.1

Do exercises 1, 3, 5, 7, 11, 13, 15, 19, 21, and 25 on page 383 of the textbook.

MODULE 9 EXERCISES

Written Assignment 10

Complete the following exercises based on assigned sections you learned in this module, and submit them to your mentor for correction and grading. Show all calculations.

1. Find the area between the curves on the given interval.

$$y = \cos x,\ y = x^2 + 2,\ 0 \le x \le 2$$

2. Find the area between the curves on the given interval.

$$y = e^{-x},\ y = x^2,\ 1 \le x \le 4$$

3. Sketch and find the area of the region determined by the intersections of the curves.

$$y = x^2 - 1,\ y = \tfrac{1}{2}x^2$$

4. Sketch and find the area of the region determined by the intersections of the curves.

$$y = \sqrt{x},\ y = x^2$$

5. Sketch and find the area of the region determined by the intersections of the curves.

$$y = \frac{2}{x^2 + 1},\ y = |x|$$

6. Sketch and estimate the area determined by the intersection of the curves.

$$y = \cos x,\ y = x^4$$

7. Sketch and estimate the area determined by the intersection of the curves.

$$y = \ln x,\ y = x^2 - 2$$

8. Sketch and find the area of the region bounded by the given curves. Choose the variable of integration so that the area is written as a single integral. Verify your answer with a basic geometric area formula.

$$y = x,\ y = 2,\ y = 6 - x,\ y = 0$$

9. Sketch and find the area of the region bounded by the given curves. Choose the variable of integration so that the area is written as a single integral.

$$x = 3y,\ x = 2 + y^2$$

10. Sketch and find the area of the region bounded by the given curves. Choose the variable of integration so that the area is written as a single integral.

$$x = y^2,\ x = 4$$

***Note:** Submitting a graph is not required; however, you are encouraged to create one for your own benefit. In your solution, simply list all relative information for the specific problem such as complete points of the form (x, y) for any local extrema, inflection points, and intercepts. If needed, give equations for asymptotes, and list any discontinuities.